

S and U-duality Constraints on IIB S-matrices

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Abstract

S and U-duality dictate that graviton scattering amplitudes in IIB superstring theory be automorphic functions on the appropriate fundamental domain which describe the inequivalent vacua of (compactified) theories. A constrained functional form of graviton scattering is proposed using Eisenstein series and their generalizations compatible with: a) two-loop supergravity, b) genus one superstring theory, c) the perturbative coupling dependence of the superstring, and d) with the unitarity structure of the massless modes. The form has a perturbative truncation in the genus expansion at a given order in the derivative expansion. Comparisons between graviton scattering S-matrices and effective actions for the first quantized superstring are made at the quantum level. Possible extended finiteness properties of maximally extended quantum supergravity theories in different dimensions is implied by the perturbative truncation of the functional form of graviton scattering in IIB superstring theory.

1 Introduction

S-duality has emerged as a self-equivalence structure of the IIB superstring and evidence has accumulated for its existence. Structures in string theories have in the past yielded a better understanding of aspects of supersymmetric field theories. In this work we shall explore a manifestly S-dual compatible, perturbative and non-perturbative, expansion of the ten-dimensional S-matrix for graviton scattering. This S-matrix is highly constrained by modular forms, and in agreement with known results in the low energy limit.

Recent work on the effective action in the low-energy limit of the superstring has revealed the modular property of the S-duality invariant graviton scattering to several orders in derivatives [1, 2], in eleven-dimensional supergravity [3, 4, 5], and for additional couplings in IIB superstring theory in [6]. We examine the same structure at all orders in derivatives and generate these terms through the use of modular forms, on the $SL(2, Z)$ fundamental domain for ten and nine-dimensional theories, and on more complicated domains for lower-dimensional compactified theories. In related work there have been $SL(2, Z)$ based reformulations at the level of the world-sheet action [7, 8, 9] which might lead to a similar description that produces higher derivative corrections compatible with S-duality of the IIB superstring as for the R^4 term [9]. Tests of S-duality at the amplitude level require knowledge of the perturbative expansion for graviton scattering at genus greater than one, and is formidable. The S-duality of the superstring imposed on the S-matrix leads to non-trivial predictions for its structure, and predicts contributions from the genus expansion without performing string perturbation theory.

Standard perturbative expansions involve expansions in the coupling constant, or in conjunction with large N ; however, Feynman diagrams are typically difficult at high loop order and alternative approaches are useful. The derivative expansion orderwise must be compatible with the non-perturbative symmetry structures that exchange weak and strong coupling and must obey these invariance properties in IIB superstring theory; possibly this re-ordering avoids summation problems, because of factorial dependence in the diagram expansion, as it is non-perturbative in the coupling and should thus take into account the solitonic contributions. Such an expansion is necessarily non-perturbative from the point of view of the microscopic coupling constant in the maximally supersymmetric theory, although perturbative in the energy scale for theories without dimensional transmutation. M-theory in the eleven dimensional limit requires such an approximation, because there is no dilatonic coupling constant in this eleven dimensional corner.

IIB superstring theory has IIB supergravity describing its zero-mode degrees of freedom. Compactification to lower dimensions through tori gives rise to several supergravity theories including the example of maximally extended $N = 8$ supergravity in four dimensions [10, 11, 12, 13]. The remnant of S-duality, and U-duality in general [14], imposes severe constraints on the perturbative expansions of the supergravity theories in various dimensions: Primarily, we are interested in the finiteness properties of the latter theories in this regard. Recently, $N = 8$ supergravity in four dimensions has been re-examined, and there is strong evidence

that the previously thought first primitive divergence of the four-point amplitude does not occur at three-loops, but at higher order [15]. In specifying the supergravity quantum theory from the low-energy limit of the superstring, a regulator must be chosen that is compatible with the global symmetries of the field equations; on the other hand, superstring theory points to a specific regulator, the one in which S-duality remains as a remnant on the massless degrees of freedom. At one-loop this is most easily seen in comparing the SUGRA expansion with the string and its non-perturbative structure.

Further divergence nullifications beyond the known properties of N=8 supergravity require a field theory mechanism. Recent work in [15] has shown that additional cancellations not accounted for in superspace powercounting arguments occur. In the explicit construction of the two-loop amplitudes these additional cancellations follow in the cut construction from use of on-shell supersymmetry Ward identities. Alternatively, this cancellation occurs because of S-duality, where this structure permeates to higher order and in various dimensions.

In this work we shall reformulate the expansion of the IIB superstring graviton scattering using constraints imposed on automorphic functions. A uniqueness theorem regarding the functional form incorporating additional properties of the scattering elements potentially allows, given S-duality, a route to computing complete S-matrix elements without perturbative string theory. A further aspect of the modular construction in terms of Eisenstein functions implies a truncation property in perturbative supergravity defined by the toroidal compactification of IIB in ten dimensions (or $d = 11$ supergravity).

This work is organized as follows. In section 2 we examine the general structure of the low-energy limit of the graviton scattering and the constraints of S-duality in the uncompactified IIB theory. In section 3 we compare the S-matrix with definitions of the quantum effective action. In section 4 we give relevant properties of the Eisenstein series used in the construction outlined in section 2. In section 5 we analyze toroidal compactifications to lower dimensions and the U-duality structure together with the $SL(2, Z)$ subgroup; the emphasis in this section is on finding further truncations. In section 6 we extract implications for the graviton scattering in the field theory limit. In the last section we give conclusions and discuss extensions related to this work.

2 S-matrices and Constraints

In Einstein frame, S-duality exchanges the coupling constant $\tau = \chi + ie^{-\phi}$ of the IIB superstring with its inverse, and more generally, under the $SL(2, Z)$ fractional linear transformation,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} . \quad (2.1)$$

An invariant perturbative expansion for the scattering of gravitons is obtained by expanding in the string scale α' at all orders in the perturbative series, rather than in the string coupling constant $g_s = e^\phi$. Such an expansion is necessarily non-perturbative in form from the point

of view of the string coupling constant, but there are constraints from the known structure of the perturbative series in supergravity to be compared with.

The low-energy expansion of the string S-matrix is found by expanding the kinematic invariants parametrizing the scattering at small values below the string scale α' . We define the Mandelstam invariants relevant for the four-point function by $s = -(k_1 + k_2)^2$, $u = -(k_1 + k_3)^2$ and $t = -(k_2 + k_3)^2$. The general structure of this expansion contains polynomial terms in the kinematic invariants together with logarithmic functions, as demanded by unitarity of the massless modes. For $s_{ij} > 4/\alpha'$, the unitarity cuts for the massive modes of the IIB superstring appear after a resummation of the former terms; an infinite resummation of the higher derivative terms may produce the required unitarity cuts for the massive string states. For example, the genus one form for a unitarity threshold of the first massive mode of the superstring is $\ln(1 - \alpha's/4)$ and may be expanded at low-energy as an infinite series in s via $\sum_{j=1}^{\infty} (\alpha's/4)^j$. The fact that the energy scale of the kinematics is below the first massive mode of the string is crucial for preserving the manifest S-duality invariant expressions so far found in the literature.

The string perturbative series is normalized with the conventions

$$\kappa_{10}^2 = \frac{1}{2}(2\pi)^7 \alpha'^4 \quad (2.2)$$

for the ten-dimensional gravitational coupling constant. Because α' enters both in the coupling from the ten-dimensional field theory point of view, as well as in the parametrization of the mass levels, disentangling of the contributions to the amplitudes from the massive modes verse the massless ones of the string needs to be carried out. However, the contributions easily separate to genus zero and one in the string perturbative series.

As of yet, there is no known consistent form of the S-duality compliant S-matrix for the IIB superstring in flat ten-dimensional Minkowski background. The S-duality invariance of the Einstein frame perturbative expansion demands a stringent form of the scattering, for example, of four-graviton scattering to low orders in the derivative expansion at fixed, but small, α' . The work of [1, 2, 16] indicates that the form of the scattering of the polynomial terms up to twelve derivatives is given non-perturbatively by the series,

$$S_{4-point} = \int d^{10}x \sqrt{g} \left[\frac{1}{\square^3} R^4 + E_{3/2}(\tau, \bar{\tau}) R^4 + E_{5/2}(\tau, \bar{\tau}) \square^2 R^4 \right], \quad (2.3)$$

in Einstein frame. $E_s(\tau, \bar{\tau})$ are non-holomorphic Eisenstein series (defined in section 4) and the IIB string coupling is $\tau = \chi + ie^{-\phi}$. We have included in the first term in (2.3) the massless bosonic exchange at tree-level. The string coupling constant is $g_s = e^\phi$ and the derivatives in the first term is shorthand (2.3) for the factor $1/stu$.

The modular construction of the graviton S-matrix is S-duality invariant, and previous attempts to generalize to all orders followed by examination of the tree-amplitude for four gravitons [17, 18, 19]. However, it gives incorrect predictions for the one-loop contribution to the fourteen derivative term $\square^3 R^4$.

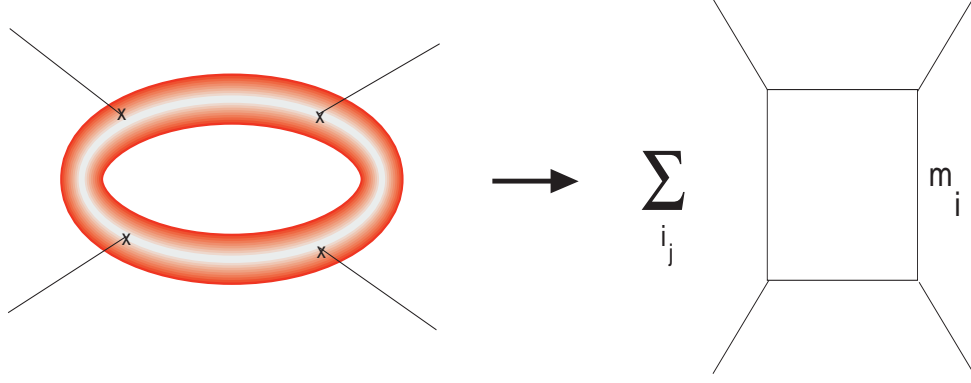


Figure 1: The low-energy limit of the four-point string genus one diagram is represented as an infinite summation of field theory diagrams with a string-inspired regulator.

The tree-amplitude for the scattering of four gravitons in a flat background in IIB theory is

$$A_{4,g=0}^{\text{IIB}} = 64R^4 \frac{e^{-2\phi}}{\alpha'^3 stu} \frac{\Gamma(1 - \frac{\alpha's}{4})\Gamma(1 - \frac{\alpha't}{4})\Gamma(1 - \frac{\alpha'u}{4})}{\Gamma(1 + \frac{\alpha's}{4})\Gamma(1 + \frac{\alpha't}{4})\Gamma(1 + \frac{\alpha'u}{4})} , \quad (2.4)$$

or in an alternative form,

$$A_{4,g=0}^{\text{IIB}} = 64e^{-2\phi} \frac{R^4}{\alpha'^3 stu} \exp\left(\sum_{p=1}^{\infty} \frac{2\zeta(2p+1)}{2p+1} \left(\frac{\alpha'}{4}\right)^{2p+1} (s^{2p+1} + t^{2p+1} + u^{2p+1})\right) , \quad (2.5)$$

where the tree-level on-shell effective action [28] found by integrating out massive string modes is contained in the second term of the expansion, i.e. $2\zeta(3)e^{-2\phi} \int d^{10}x \sqrt{g} R^4$. The R^4 factor represents the well-known tensor [20], the square of the Bel-Robinson tensor [21], which in linearized k -space form is found by contracting eight momenta with the external four polarization vectors,

$$R^4 = t_8^{\mu_1 \dots \mu_8} t_8^{\nu_1 \dots \nu_8} \prod_{i=1}^4 \varepsilon_{\mu_i \nu_i} k_{\mu_{i+4}}^i k_{\nu_{i+4}}^i , \quad (2.6)$$

in momentum space, or in the linear approximation (contributing only to the four-point function) of the contraction of four Weyl tensors with the tensors t_8 .

The conjecture in [18] for the non-perturbative form of the S-matrix is found by replacing the Riemann-zeta functions in (2.5) with an Eisenstein series through,

$$\zeta(2p+1) \rightarrow E_{p+1/2}(\tau, \bar{\tau}) , \quad (2.7)$$

which takes into account the Einstein frame dependence of the string scattering. Although this substitution leads to the correct R^4 coupling, it predicts a contribution at genus one for the $\square^3 R^4$ term that is *twice* its calculated value and is not normalization dependent. It also does not address the required unitarity properties of the massless modes (i.e. supergravity). Building modular invariant functions from covariantizing through the substitution in (2.7) is not unique, and we shall modify this conjecture in this work.

The one-loop amplitude for four-graviton scattering is

$$A_{4,g=1}^{\text{IIB}} = R^4 \int \prod_{i=1}^4 d^2 z_i \int_{\mathcal{F}_1} \frac{d^2 \tau}{\tau_2^2} \prod_{i \neq j} |E(z_i, z_j, \tau)|^{-\alpha' s_{ij}} e^{-\frac{2\pi}{\tau_2} k_i \cdot k_j \text{Im}(z_i) \text{Im}(z_j)} , \quad (2.8)$$

with the genus one prime form,

$$E(z = z_i - z_j; \tau) = \frac{\Theta\left[\frac{1}{2}\right]\left(z, \tau\right)}{\Theta'\left[\frac{1}{2}\right](0, \tau)} , \quad (2.9)$$

and may be similarly expanded in the $\alpha' \rightarrow 0$ limit [5]. (The denominator in (2.9) drops out due to momentum conservation.) The low-energy expansion and analytic properties of this amplitude is taken in [22, 23, 5], and the integral expansion may be directly reproduce as an infinite summation of field theory ten-dimensional box diagrams with the appropriate internal mass parameters as in figure 4. In [5] the expansion in α' is carried out to order $\square^2 R^4$ in the derivative (or equivalently α') expansion. We shall need to disentangle the massive from the massless mode contributions in this expansion to compare with the predicted form of graviton scattering in pure IIB supergravity. The massive modes generate terms in the q expansion of the oscillators when the power is non-zero. It is not that the fundamental integration region in (2.8) is demanded by a modular cancellation between the massive and massless modes of the superstring; modular $SL(2, Z)$ transformations of the punctured torus map regions in the field theory limit of the loop integration to other regions in addition to those of the massive contributions. The string, in the field theory limit, dictates a well-defined regulator. In the open-string limit suitable for Yang-Mills theory, this complication does not enter until the world-sheet corresponds to two-loops. Although straightforward at one-loop, technicalities associated with modular parameterizations and world-sheet ghosts at higher genus complicate a similar expansion at this order and will be analyzed in future work.

We end this section with a clarification of the string-inspired regulator at one-loop. The fundamental domain \mathcal{F}_1 is over a restricted domain in the complex plane with a non-trivial region near the origin,

$$\mathcal{F}_1 = \left\{ \tau = \tau_1 + i\tau_2 : \tau_1^2 + \tau_2^2 \geq 1 , |\tau_1| \leq \frac{1}{2} \right\} . \quad (2.10)$$

In field theory the box diagram with masses on the internal lines labelled by j is given by the integral representation,

$$\begin{aligned}
I_4(k_j, m_j) &= \int \frac{d^d l}{(2\pi)^d} \prod_j \frac{1}{(l - p_j)^2} \\
&= (2\pi)^{-\frac{d}{2}} \int \prod_{i=1}^4 da_i \delta(1 - \sum_{j=1}^4 a_j) \int_0^\infty \frac{d\tilde{\tau}_2}{\tilde{\tau}_2^{-2+d/2}} e^{-\tilde{\tau}_2 [f(a_i; p_i) + a_j m_j^2]} \\
&= {}_R (2\pi)^{-\frac{d}{2}} \Lambda^{3-d/2} \int \prod_{i=1}^4 da_i \delta(1 - \sum_{j=1}^4 a_j) \int_1^\infty \frac{d\tau_2}{\tau_2^{-2+d/2}} e^{-\tau_2 [f(a_i; p_i) + a_j m_j^2]/\Lambda^2} \quad (2.11)
\end{aligned}$$

where in the last line we have implemented a Schwinger proper time regulator. The boundary on the integration is specified through the use of a regulator. In a Schwinger proper time regulator the region for $\tilde{\tau}_2$ is be limited to $\tilde{\tau}_2 \geq 1/\Lambda^2$. In dimensional reduction (or regularization) we need to integrate over the entire region of $\tau_2 \geq 0$ with the dimensionally continued integral (in the measure). However, an alternative cutoff scheme for the supergravity amplitudes that is consistent with the underlying modular invariance of the string is to use the region delimited in (2.10), which is at one-loop found by replacing the integration regime in (2.11),

$$\int_1^\infty \frac{d\tau_2}{\tau_2^2} \rightarrow \int_{\mathcal{F}_1} \frac{d^2 \tau}{\tau_2^2} . \quad (2.12)$$

The region in the complex plane with $\tau_2 \geq 1$ is the same, because the integration over τ_1 integrates to unity in (2.11). Its difference from the Schwinger proper time cutoff is in a finite region in the corner of moduli space. The integration region in (2.12) should be considered as an alternative regularization scheme, one that is compatible in supergravity with the non-perturbative S-duality transformations of the superstring.

The parameterization of the domain of integration in the higher-genus moduli space [24] (and references) serves as a string-inspired regulator for higher loop field theory, and gives rise to the string-inspired regulated supergravity quantum field theory. The regulated higher genus form is similar to (2.12) but with a more complicated corner region. The Schwinger proper-time regularization generalizes through the additional moduli of the superstring as at one-loop.

2.1 Ten Dimensions

The previously conjectured form [18] for the manifestly S-duality invariant function corresponding to the four-graviton scattering amplitude used the Eisenstein functions $E_s(\tau, \bar{\tau})$ to relate the tree-level contributions to the higher genus ones by covariantizing the phase in (2.5). However, it clearly is only an approximation for the reason that it predicts the incorrect coefficient of the $\square^3 R^4$ term at genus one, and does not take into account the thresholds

associated with the massless string modes (i.e. $d = 10$ IIB supergravity). In this section, we give a different version that incorporates both of these features.

The simplest modification is to enlarge the set of Eisenstein functions to their generalized non-holomorphic and non-modular invariant counterparts. The constraints for writing down suitable modular forms and combinations follow from : 1) the perturbative series is a power series in even powers of the string coupling τ_2 , 2) invariance under modular transformations in Einstein frame, 3) compatibility with the unitarity structure of the massless string modes, 4) the power series predicts the appropriate maximum power of τ_2 consistent with the derivative term $\square^k R^4$ term in Einstein frame at tree-level. We shall also consider only the Eisenstein series with half-integer or integer characteristics (the value of which is related to the half-integral R-charge shown in [6] and [25]) and in ten dimensions those that converge (lower dimensional compactifications of supergravity have infra-red divergences in $d \leq 6$, and possibly the convergence properties are related to the infra-red divergences). The first condition is tight enough to rule out most commonly known modular forms, and we shall consider the set of generalized Eisenstein functions for $SL(2, Z)$, $E_s^{(q, -q)}(\tau, \bar{\tau})$,

$$E_s^{(q, -q)}(\tau, \bar{\tau}) = \sum_{(p, q) \neq (0, 0)} \frac{\tau_2^s}{(p + q\tau)^{s+q}(p + q\bar{\tau})^{s-q}} . \quad (2.13)$$

Their properties are discussed in section 4, together with the exceptional modular invariant function

$$f = \ln \tau_2 |\eta(\tau, \bar{\tau})|^4 \quad (2.14)$$

which is related to the non-convergent $E_1(\tau, \bar{\tau})$ series after subtracting the singularity, denoted by $\hat{E}_1(\tau, \bar{\tau})$,

$$\hat{E}_1(\tau, \bar{\tau}) = -\pi \ln(\tau_2 |\eta(\tau)|^4) . \quad (2.15)$$

The $SL(2, Z)$ Laplacian acting on $f(\tau, \bar{\tau})$ is one,

$$\tau_2^2 \partial_\tau \partial_{\bar{\tau}} \ln \tau_2 |\eta(\tau, \bar{\tau})|^4 = 1 , \quad (2.16)$$

which in the string setting represents contributions from only one perturbative order (the form is required for genus one and for the $\square \ln \square R^4$ tensor). Our ansatz consists of polynomials in this combined set. The cusp forms are modular invariant functions that vanish at $\tau_2 \rightarrow \infty$ and admit the expansion,

$$f_{\text{cusp}}(\tau, \bar{\tau}) = \sum_{n \neq 0} a_n \tau_2^{\frac{1}{2}} K_{n-1/2}(2\pi |n| \tau_2) e^{2\pi i n \tau_1} , \quad (2.17)$$

expressed in terms of exponentials of τ_2 . They do not contribute to the perturbative expansion of either the superstring or supergravity, and we shall not consider them or the modifications of the polynomial terms by them in any detail. The first instanton correction

to the R^4 term in [1] provides evidence that they do not contribute to this order, and a similar calculation at higher α' is necessary to predict whether or not they contribute to the higher derivative terms. Explicit forms for the cusp forms on the fundamental domain $U(1)\backslash SL(2, R)/SL(2, Z)$ are not known [30]. Similarly, the ones relevant for the moduli spaces of the toroidally compactified theories are not known (these spaces contain the former as a subspace).

We first discuss the polynomial terms arising in the low-energy expansion, followed by the non-analytic (logarithmic) ones. The transformation from string frame to Einstein frame, $\eta_{\mu\nu}^{(s)} = \tau_2^{\frac{1}{2}} \eta_{\mu\nu}^{(E)}$, induces a coupling constant into the derivative expansion that is important for the τ_2 counting. For example, the $\square^k R^4$ terms in the string frame at tree-level are described in Einstein frame,

$$A_{4,g=0}^{IIB,k} = N_{g=0}^k \zeta\left(\frac{3}{2} + \frac{k}{2}\right) \tau_2^{3/2+k/2} \square_{(E)}^k R_E^4, \quad (2.18)$$

for $k \neq 1$, and where the $e^{-2\phi}$ string coupling inherit in (2.5) and the transformation of the tensors are taken into account. This power of τ_2 is the maximum one possible in the perturbative series and successive higher genus contributions lower it by two units successively. The general form of the contribution at arbitrary k to all genus,

$$A_4^{IIB,k} = f_k(\tau, \bar{\tau}) \square_{(E)}^k R_{(E)}^4, \quad (2.19)$$

demands the structure of $f_k(\tau, \bar{\tau})$ to have the form,

$$f_k(\tau, \bar{\tau}) = \zeta\left(\frac{3+k}{2}\right) \tau_2^{\frac{3}{2}+\frac{k}{2}} + a_1 \tau_2^{-\frac{1}{2}+\frac{k}{2}} + a_2 \tau_2^{-\frac{3}{2}+\frac{k}{2}} + \dots + \mathcal{O}(e^{2\pi i \tau}). \quad (2.20)$$

The coefficients a_g are constants that may be found by doing explicit low-energy string perturbation theory calculations up to genus g . We will see that given the set of modular forms discussed above the series a_k receives non-vanishing values up to a maximum genus for a given k .

The fact that only one tensor, namely the eight-derivative R^4 covariantized from the linear form in (2.6), appears in the description of the four-point function is consistent both with perturbative IIB superstring theory up to genus four [26] as well as with constraints in supergravity from unitarity constructions. This property is due to $N = 8$ supersymmetry at one [20] and two-loops and also to higher orders [15]: The uniqueness of this tensor structure is the primary influence of $N = 8$ supersymmetry in constraining the S-matrix as the loop integrations are not constrained directly by supersymmetry without additional regulator specification (such as choosing one compatible with the global duality symmetries of the classical field equations of maximal supergravity). Lower derivative tensor structures appear in theories with lower supersymmetric gravitational theories [27].

For $k = 0$, the only function in the set considered that could describe the expansion is $E_{3/2}^{(0,0)}$ (modulo cusp forms); the R^4 term then receives perturbative contributions at tree-level and one-loop. Explicit calculations in supergravity indicate that, at two-loops and higher, an

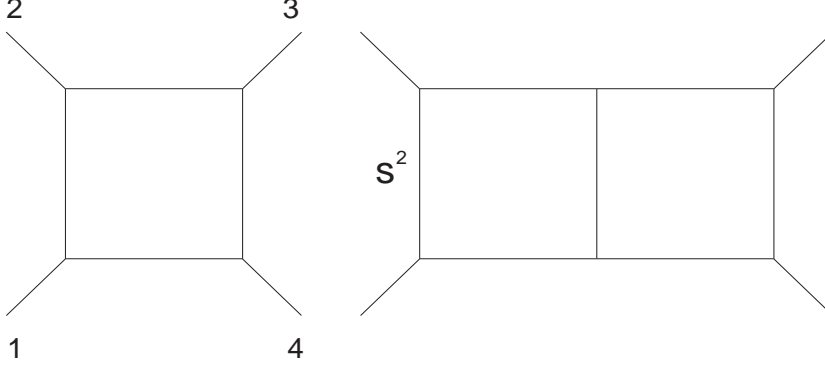


Figure 2: Sample supergravity field theory contributions to the one- and two-loop graviton scattering before symmetrization. To these orders the derivatives are explicitly extracted from the tensor integrations.

additional \square^2 may explicitly be pulled out from within the loop integrations (in figure 2 we list the integral contributions, where the tensor R^4 is not displayed). Thus, this truncation is consistent with known results (at tree and one-loop the massive modes in the q expansion explicitly are of order α' , hence \square by dimensional analysis, higher than the massless modes). The value $k = 1$ is special because momentum conservation of the four external gravitons, through $s + t + u = 0$, forces the on-shell ten-derivative polynomial term to vanish. At $k = 2$, again there is only one function one may write down, namely $E_{5/2}(\tau, \bar{\tau})$, and predicts a genus zero and genus two contribution but none from genus one; explicit string one-loop calculations shows the vanishing of the genus one coefficient.

At values of $k \geq 3$ a break with the previous pattern emerges because the non-modular invariant $E_s^{(q, -q)}(\tau, \bar{\tau})$ for $q \neq 0$ may contribute. At $k = 3$ we may introduce the form

$$f_3(\tau, \bar{\tau}) = \alpha_1 E_{3/2}^2(\tau, \bar{\tau}) + \alpha_2 E_{3/2}^{(1, -1)}(\tau, \bar{\tau}) E_{3/2}^{(-1, 1)}(\tau, \bar{\tau}) , \quad (2.21)$$

(where $E_{3/2}^{(-1, 1)} = E_{3/2}^{*(1, -1)}$). It has the large τ_2 perturbative expansion,

$$f_3(\tau, \bar{\tau}) = 4(\alpha_1 + \alpha_2) \left[\zeta(3)^2 \tau_2^3 + \left(\frac{\alpha_1 - \alpha_2/3}{\alpha_1 + \alpha_2} \right) 4\zeta(2)\zeta(3)\tau_2 + \left(\frac{\alpha_1 - \alpha_2/9}{\alpha_1 + \alpha_2} \right) 4\zeta^2(2)\tau_2^{-1} \right] \quad (2.22)$$

The relative coefficient between α_1 and α_2 is chosen to agree with the genus one contribution for $\square^3 R^4$. Note that (2.22) differs from previous forms because of the introduction of the modular invariant contribution of the generalized Eisenstein functions. A relative value of $\alpha_1 = \frac{5}{3}\alpha_2$ leads to agreement with the calculated genus zero and one coefficients in string theory; this combination predicts a non-vanishing coefficient at genus two and *no further contributions*.

Table 1: The marks indicate which modular forms could contribute at genus g to the $\square^4 R^4$ term in the four graviton scattering amplitude.

	$g = 0$	$g = 1$	$g = 2$	$g = 3$
α_1	\checkmark			\checkmark
α_2		\checkmark	\checkmark	

Table 2: Example contributions to $\square^k R^4$ arising in string perturbation theory at genus g from the structure of the generalized Eisenstein series.

	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	\dots
R^4	\checkmark	\checkmark				
$\square R^4$						
$\square^2 R^4$	\checkmark		\checkmark			
$\square^3 R^4$	\checkmark	\checkmark	\checkmark			
$\square^4 R^4$	\checkmark	\checkmark	\checkmark	\checkmark		
$\square^5 R^4$	\checkmark	\checkmark	\checkmark	\checkmark		
\dots						

Combinations of these modular forms is straightforward to construct at higher derivatives, although it is not uniquely determined through direct covariantization via the substitution in (2.7). Furthermore, in contrast to the $k \leq 3$ cases, such combinations allow for perturbative contributions arising from modular forms that contribute at orders of genus solely above tree-level. The non-analytic terms in the low-energy expansion require such combinations.

As another example, consider $k = 4$ and $k = 5$. In this case we may use,

$$f_4(\tau, \bar{\tau}) = \alpha_1 E_{7/2}(\tau, \bar{\tau}) + \alpha_2 E_{3/2}(\tau, \bar{\tau}) , \quad (2.23)$$

and,

$$\begin{aligned} f_5(\tau, \bar{\tau}) = & \beta_1 E_2(\tau, \bar{\tau})E_2(\tau, \bar{\tau}) + \beta_2 E_2(\tau, \bar{\tau}) + \beta_3 E_{3/2}(\tau, \bar{\tau})E_{5/2}(\tau, \bar{\tau}) \\ & + \beta_4 E_2^{(1,-1)}(\tau, \bar{\tau})E_2^{(-1,1)}(\tau, \bar{\tau}) + \beta_5 E_{3/2}^{(1,-1)}(\tau, \bar{\tau})E_{5/2}^{(-1,1)}(\tau, \bar{\tau}) + \text{c.c.} . \end{aligned} \quad (2.24)$$

There is one condition on coefficients $\beta_1, \beta_2, \beta_4$ in the asymptotic expansion from the perturbative expansion, that the τ_2^{-1} term is zero. The number of different combinations increases with larger k values. Similar functions may be constructed to all orders, and the functional form at higher derivatives is one of the central results in this work. In (2.23) the different combinations generate perturbative corrections at the following genus orders tabulated in Table 1.

The perturbative series of the polynomial terms for the $\square^k R^4$ tensor in string theory, using the generalized Eisenstein functions in (2.13) for $s = n/2$ and n integer in a polynomial fashion, receives corrections up to maximum $g_{\max} = \frac{1}{2}(k+2)$ genus in string theory for k even and $g_{\max} = \frac{1}{2}(k+1)$ genus for k odd, listed in Table 2. Higher genus calculations in IIB superstring theory is necessary to verify the form as well as to make agreement with the coefficients; an indirect resummation of the leading terms in τ_2 of the derivatives must also agree with the thresholds of the massive modes of the superstring and might fix the coefficients.

In the remainder of this section we examine other functions built out of Eisenstein series. Generic combinations of the generalized Eisenstein series either do not give rise to the appropriate powers of τ_2 in accord with the string perturbative series or have values of $s \leq 1$ and not converge. Ratios of Eisenstein series in the large τ_2 perturbative regime, for example,

$$\frac{E_3(\tau, \bar{\tau})}{E_{\frac{3}{2}}(\tau, \bar{\tau})} = \frac{a_3 \tau_2^3 + b_3 \tau_2^{-2} + n_1 e^{-2\pi\tau_2} + \dots}{a_{3/2} \tau_2^{\frac{3}{2}} + b_{3/2} \tau_2^{-\frac{1}{2}} + m_1 e^{-2\pi\tau_2} + \dots}, \quad (2.25)$$

either do not generically reflect the appropriate τ_2 dependence of string perturbation theory, or else,

$$\frac{E_{3/2}^3(\tau, \bar{\tau})}{E_{5/2}(\tau, \bar{\tau})} = \frac{(a_{3/2} \tau_2^{\frac{3}{2}} + b_{3/2} \tau_2^{-\frac{1}{2}} + e^{2\pi i\tau} + \dots)^3}{a_{5/2} \tau_2^{\frac{5}{2}} + b_{5/2} \tau_2^{-\frac{3}{2}} + c e^{2\pi i\tau} + \dots} \quad (2.26)$$

generate at large τ_2 perturbative τ_2 dependence in accord with string perturbation theory and could potentially be used to describe the $\square^5 R^4$ term. However, the dependence of τ_2 on the exponential terms generically does not match with expectations of the D-instanton series (which begin with a single order one factor for the first correction after expanding (2.26)). The function in (2.26) has dependence on half-integral powers of τ_2 , and its exponentials are suppressed by a power of $\tau_2^{5/2}$. Rigorously, we can not rule out the ratio combinations similar to that in (2.26) at all orders without further constraints.

Non-half integral or integral values of s also generically lead to dependence on τ_2 which is not captured by perturbation theory. For example, the product

$$\begin{aligned} E_{5/4}^2(\tau, \bar{\tau}) &= (a \tau_2^{\frac{5}{4}} + b \tau_2^{-\frac{1}{4}} + \mathcal{O}(e^{2\pi i\tau}))^2 \\ &= a^2 \tau_2^{\frac{5}{2}} + 2ab \tau_2 + b^2 \tau_2^{-\frac{1}{2}} + \mathcal{O}(e^{2\pi i\tau}), \end{aligned} \quad (2.27)$$

has a τ_2 factor which is not suppressed by a factor of τ_2^2 relative to the first term and cannot arise in string perturbation theory. Generic products of Eisenstein series with non-half integral values of s also have τ_2 dependence that does not agree with the perturbative series.

Any of the functions built out of the products of generalized Eisenstein series may also be used to generate further automorphic functions by acting on them with the covariantized

Laplacian $\nabla = \tau_2^2 \partial_\tau \partial_{\bar{\tau}}$, e.g. $\nabla E_{3/2}^2(\tau, \bar{\tau})$. The polynomial action of ∇ generates the same perturbative truncation in the modular ansatz up to a maximum genus for a given α' order, although the instanton corrections may be modified together with cusp forms. Their effect on the perturbative series is to adjust coefficients in the perturbative series up to the maximum genus. The action up to \square^3 order in the derivative expansion does not introduce additional automorphic functions because the single Eisenstein functions are eigenfunctions of ∇ , and their use at higher derivatives appears redundant perturbatively.

In lower dimensional toroidally compactified theories, the limiting Eisenstein functions relevant for describing the $\square^k R^4$ terms have less well behaved convergence and are possibly related to the infra-red divergences in $d \leq 6$ supergravity. The general structure of the perturbative series arising through the modular form construction required by S-duality invariance is listed in table 2.

The pre-factors $E_{3/2}(\tau, \bar{\tau})$ and $E_{5/2}(\tau, \bar{\tau})$ of the low-energy polynomial expansion up to $\square^2 R^4$ satisfy Laplacian eigenvalue conditions. We are not imposing a property

$$4\tau_2^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \bar{\tau}} f_k^{(i)}(\tau, \bar{\tau}) = \lambda_k^{(i)} f_k^{(i)}(\tau, \bar{\tau}) , \quad (2.28)$$

on terms $f_k^{(i)}$ of $f_k(\tau, \bar{\tau})$ or a similar one on the potentially covariantized phase in (2.5), which together with the asymptotic behavior in (2.20), limits only single Eisenstein series and their generalizations to being a solution (together with cusp forms) to (2.28) [30]. The $SL(2, Z)$ covariantization of the leading asymptotic term in f_k does satisfy a Laplacian condition, i.e. $\nabla(\tau_2^{\frac{3+k}{2}}) = 4(\frac{3+k}{2})(\frac{1+k}{2})(\tau_2^{\frac{3+k}{2}})$, but may be covariantized with $SL(2, Z)$ in different ways. For example, the simplest gives $E_{\frac{3}{2}+\frac{k}{2}}(\tau, \bar{\tau})$ but also $E_{\frac{3}{4}+\frac{k}{4}}^2(\tau, \bar{\tau})$ or $|E_{\frac{3}{4}+\frac{k}{4}}^{(1,-1)}(\tau, \bar{\tau})|^2$. A naive covariantization of the leading terms is not unique and for this reason the graviton scattering then is not going to exponentiate directly into the form of a tree-like amplitude.

2.2 Non-analytic terms

The unitarity structure of the perturbative superstring amplitudes also requires a description in terms of modular forms in order to construct manifestly S-dual scattering. Clearly such a construction is different than the above because at tree-level there are no non-analytic contributions and the perturbative expansion involving logarithmic functions, for example $\ln^L(\square)$ and $\ln \ln \dots \ln \square$ (up to L iteratively), begins at genus one.

The form of the action containing higher orders in derivatives depends on the choice of the definition of the effective action but is uniquely defined by an S-matrix computation in string theory. As required by unitarity, the S-matrix and the effective action will both contain logarithmic terms. At one-loop, for example, explicit expressions found from expanding the integrated four-point S-matrix element has a contribution of the form

$$C_{\log}^{g=1} = (s \ln s + t \ln t + u \ln u) R^4. \quad (2.29)$$

Higher logarithms appear at multi-loop and similar structure is required in higher-point gravitational amplitudes. Under the transformation to Einstein frame from string frame the kinematic invariants pick up factors $s_s \rightarrow s_e \sqrt{\tau_2}$. (The metric is rescaled by a square-root factor of the coupling constant in ten dimensions.) The logs potentially produce further non-analyticity in the string coupling constant,

$$s \ln s \rightarrow \sqrt{\tau_2} s_e \ln(\sqrt{\tau_2} s_e). \quad (2.30)$$

In Einstein frame, the coupling $e^{-\phi} = \tau_2$ does not just count loops via the the topological coupling,

$$S_{\text{dil}} = \int d^2 z \sqrt{g} R \langle \phi \rangle = 2(1 - g) \phi, \quad (2.31)$$

but rather includes a dilatonic factor within the free superstring action. S-duality does not simply exchange weak coupling with strong coupling in the Einstein frame for individual modes of the string: In a field theory setting the the coupling constant is inverted, but also the contributions of the massive modes of the superstring get mixed differently than the massless ones within the integration because of the scale factor introduced in transforming to Einstein frame. The τ_2 dependence in the massive modes is not removed in the space-time propagator $\sqrt{\tau_2} s_e - m_i^2$.

In (2.29) because of momentum conservation, the τ_2 dependence in the logarithm cancels out ($s + t + u = 0$). The unitarity construction below is independent of τ_2 dependence in the non-analytic (i.e. logarithmic) terms at higher loops, which iteratively constructs $SL(2, Z)$ invariant non-analytic terms.

The on-shell polynomial terms in previous sections are sufficient to determine, through supersymmetrizing the *on-shell* $\square^k R^4$ tensor, these non-analytic terms through a unitarity construction. The imaginary part in a particular channel, for example $s = -(k_1 + k_2)^2$, of the four-graviton scattering amplitude is determined through the unitarity relation

$$\begin{aligned} \text{Im}_s A_4(k_i) = \sum_{n=2}^{\infty} \sum_{\lambda_j} \int d\phi_n A_{n,\lambda_j} \left[g(k_1), g(k_2); p(\tilde{k}_1), p(\tilde{k}_2), \dots, p(\tilde{k}_n) \right] \\ \times A_{n,\lambda_j}^* \left[g(k_3), g(k_4); p(\tilde{k}_1), p(\tilde{k}_4), \dots, p(\tilde{k}_n) \right], \end{aligned} \quad (2.32)$$

with the n -body phase space integration measure given by,

$$d\phi_n = \prod_{j=1}^n \frac{d^d \tilde{k}_j}{(2\pi)^d} \delta\left(\sum_{i=1}^n \tilde{k}_i + \sum_{j=1}^4 k_j\right) \prod_{j=1}^n \delta^{(d)}(\tilde{k}_j^2) \Theta(\tilde{k}_j^0), \quad (2.33)$$

and where λ_j denotes the quantum numbers of the physical states “ p ” of the intermediate lines (gravitons and their supersymmetric partners in the gravitational multiplet). The phase

space integration is over the region

$$\tilde{k}_j^2 = 0 \quad \sum_{i=1}^4 k_i + \sum_{j=1}^n \tilde{k}_j = 0 , \quad (2.34)$$

of out-going momenta. The equation in (2.32) gives an iterative construction of the non-analytic terms in the four-graviton scattering amplitude. At energies $s \leq 4/\alpha'$ only the massless modes of the superstring contribute to the imaginary part in the s-channel. Furthermore, the *on-shell* supersymmetrization of the $\square^k R^4$ terms (together with a similar construction of higher-point graviton scattering amplitudes) provides the necessary amplitudes A_n that are to be inserted into (2.32); a on-shell linearized IIB superspace is known [31] in the absence of an off-shell one which might aid in the explicit supersymmetrization to higher order. Given an S-duality compliant form of the higher derivative polynomial terms, the non-analytic ones are necessarily also invariant under $SL(2, Z)$ transformations through (2.32). Higher-point graviton scattering amplitudes may also be S-duality covariantized which is an ingredient in (2.32).

We may expand the amplitudes in (2.32) to a total order m which receives contributions from the respective expansions of the two amplitudes to all orders k and l so that $k + l = m$. The order α'^m non-analytic term is then determined from the polynomial ones together with the lower-order non-analytic ones. In this iterative manner, the non-analytic terms in the low-energy expansion of the gravitational S-matrix is $SL(2, Z)$ invariant if the polynomial terms are.

3 Effective action and S-matrix in derivative expansion

We shall re-examine the eight-derivative R^4 term in this section to review its form in view of the S-matrix; the off-shell effective action for the IIB superstring is difficult to define because off-shell string scattering and the action for the five-form self-dual field strength off-shell are not available directly although several different on-shell effective action constructions may be given. In this section we examine primarily the four-point function, the form of which does not alter significantly at one-loop; however, at higher-point because of contributing independent high-point diagrams from boxes to n -gons ($n \leq d$) there will be significant differences at genus one and higher.

The on-shell effective action presented in the literature up to eight derivatives [28, 1] is virtually indistinguishable from the S-matrix due to supersymmetry. There are two differences between different functional form of an on-shell effective action to this derivative order. First, the term arising from massless string exchange at tree-level, i.e. in $\alpha' \rightarrow 0$ limit the graph in figure 3 with intermediate boson lines, is not included. Second, there is regulator dependence in the quantum one-loop contribution in the field theory setting (entirely from the massless modes in a q -expansion) that breaks the S-duality invariance if a string-inspired regulator is not used in comparing the supergravity with the string.

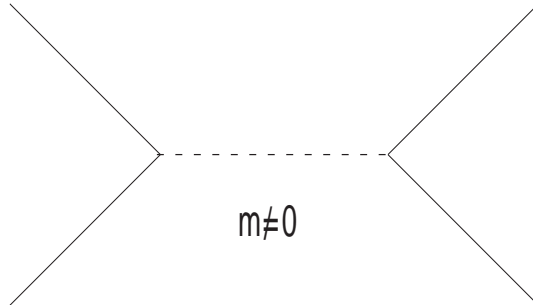


Figure 3: The diagrams with intermediate massive string modes contributing to the tree-level graviton exchange.

Furthermore, the linearized $N = 8$ (non-linear) supersymmetry will cancel any triangle or bubble sub-graph in a multi-loop graviton scattering amplitude. The string diagrams then, in the field theory limit, that have been included in the definition of the IIB effective action are identical to the ones that contribute to the four-point scattering amplitude in IIB superstring theory. Although at the four-point level the only difference between the S-matrix and the effective action in [1] is contained in (3.35), at higher-point there are contributions of external massless trees to non-vanishing loop diagrams and then more differences associated with the external massless trees from a definition adopted in [29]. Agreement with results in [6] is obtained in an S-matrix calculation in superstring theory.

In this remainder we examine the differences of the different definitions together with S-duality at the eight derivative order. In Einstein frame, the massless tree-exchange, identical to that in figure 3 but with intermediate bosonic modes together with the four-point vertex, gives rise to a covariantized contribution,

$$A = \frac{1}{\kappa_{10}^2} \int d^{10}x \sqrt{g} \frac{1}{\square^3} R^4, \quad (3.35)$$

where the \square^3 represents the appropriate combination of derivatives to produce a $1/stu$ in momentum space, and is independently invariant under the S-duality transformation. In not keeping it, the effective action is thought of as giving rise to 1PI quantum corrected vertices which must be sewn together to generate an S-matrix, i.e. illustrated as shaded circles in figure 3, and requires an off-shell generalization to construct the S-matrix. Furthermore, at one-loop there is an additional $SL(2, Z)$ of modular invariance and in not keeping the term in (3.35) arising from the massless sector breaks this in the string scattering perturbation theory. To this order the terms that break modular invariance do not break with S-duality, however, at higher-order this is possible.

We examine a more precise definition for the S-duality invariant graviton scattering expression in the following - different definitions of the low-energy quantum effective action

will produce further differences at the four-point level at order eight derivatives than just that in (3.35) and potentially break the S-duality. The S-duality invariant expressions that have been obtained so far demand that the external kinematics are below the string scale, e.g. $s_{ij} \leq 4/\alpha'$ despite the fact that the massless and massive modes are being integrated out at the quantum level. Not keeping the term in (3.35) gives a definition similar to a combination of a one-particle irreducible for the massless quantum fields together with one-particle reducible for the massive modes. In examining the implications of S- and U-duality on the supergravity theory the regulator must be chosen in a way that is compatible with these symmetries of the classical field equations.

Supersymmetry together with M-theory constraints suggest that the coefficient of the R^4 term is a function f satisfying a Laplacian condition on the fundamental domain of $U(1)\backslash SL(2, R)/SL(2, Z)$: $\tau_2^2 \partial_\tau \partial_{\bar\tau} f = 3/4 f$. Boundary conditions coming from the perturbative one-loop calculation need to be specified in order to find the solution. In supergravity the domain of vacua is the entire complex τ -plane of couplings as S-duality is not a structure only in the massless sector of the string only.

The general form of the tree- and one-loop contributions to the R^4 term are of the type,

$$f_{\text{pert}}(\tau, \bar\tau) = a\tau_2^{3/2} + b\tau_2^{-1/2} , \quad (3.36)$$

and arise from massive mode exchange at tree-level and massless ones at one-loop (with coefficients a and b respectively). The $SL(2, Z)$ invariant completion of the former term gives a form,

$$f(\tau, \bar\tau) = a\tau_2^{-1/2} + b \sum_{p,q} \frac{\tau_2^{3/2}}{|p + q\tau|^3} \quad (3.37)$$

and agrees with perturbative IIB superstring theory when,

$$f = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + \mathcal{O}(e^{2\pi i\tau}) , \quad (3.38)$$

where the exponentially suppressed terms correspond to k -multiple D-instanton corrections in the superstring theory (for example, calculated for $k = 1$ in [1]). The effective actions found by distinguishing the massless modes of the superstring with different treatments changes the first coefficient a in (3.37) to different values. It measures the regulator influence in the supergravity, that is, the cutoff Λ dependence or the use of dimensional regularization. Only for one value is the result in (3.37) $SL(2, Z)$ invariant, $a = 0$, and that is the one that comes directly from either an S-matrix element calculation in IIB string theory or through supergravity with a regulator modelling the one that arises in the low-energy limit of the quantum superstring.

In supergravity at one-loop the massless modes contribute,

$$I(s, t, u) = \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2(l - k_1)^2(l - k_1 - k_2)^2(l + k_4)^2} + u \leftrightarrow t + s \leftrightarrow u , \quad (3.39)$$

to the R^4 tensor and must be regulated because it is quadratically divergent in ten dimensions. In dimensional reduction ($d = 10 - \epsilon$) the result is,

$$I(s, t, u) = C_\epsilon \left[\frac{1}{\epsilon}(s+t) + \frac{1}{\epsilon}(u+s) + \frac{1}{\epsilon}(t+u) \right] + (\text{non-analytic}) , \quad (3.40)$$

and gives no contribution to R^4 ; dimensional reduction breaks S-duality. However, in a string-inspired regulator (2.12) the result from (3.39) is

$$A_4^{[N=8]}(k_i, \epsilon_i) \sim R^4 \left[\Lambda^2 + (s^2 + t^2 + u^2) \ln \Lambda^2 + (\text{finite}) \right] . \quad (3.41)$$

The quadratic divergence in a string-inspired regulator is roughly proportional to the inherit string scale $\Lambda^2 \sim 1/\alpha'$. The precise coefficient of the one-loop contribution to the R^4 term in d -dimensions depends on the regulator chosen and agrees with the string result for the R^4 term when the domain in (2.12) is taken. Scaling the coupling constants to force agreement between a general integration in (3.41) changes the instanton corrections predicted from the factor $E_{3/2}$ of the R^4 term. The calculation may be straightforwardly generalized to arbitrary dimensions and to its ultra-violet finite loop integration when $d < 8$.

The string-inspired regulator from the field theory point of view makes the calculated result from (3.41) agree with S-duality. Changing the regulator with the use of dimensional reduction gives a different value, $c = 0$ in (3.38) from (3.40). The above regulators preserve (non-linear) supersymmetry to this order, but only one gives rise to an explicit $SL(2, Z)$ invariant expression, namely the string-inspired regulator built into the supergravity theory.

S-duality is not expected to be a property of the low-energy IIB supergravity theory, as may be explicitly found to the order eight derivative R^4 term by dropping the massive mode contributions which contribute the $\tau_2^{3/2}$ term in (3.37). However, as a structure in the IIB superstring, it survives as a remnant in the IIB supergravity theory directly if a regulator is chosen so that the integration region in a first quantized form is chosen to mimic that of the superstring. At one-loop this corresponds to using a Schwinger proper time form of the four-point function in (2.12).

There is a one parameter family of functions $f_{\text{pert}}(\tau, \bar{\tau})$ consistent with supersymmetry that leads to the form in (3.37), and the definition that arises from the low-energy limit of the string amplitude or in supergravity with the regulator chosen in (2.12) agrees with the S-duality invariant form when $a = 0$ in (3.37). At two-loops the supergravity modular parameterization and regulator has been elucidated in [5]. Explicit calculations of one-loop four-graviton scattering in IIB supergravity indicate that all the different definitions of the effective action are encoded in one term to order eight derivatives.

In deducing results regarding the perturbative series of maximal supergravity directly from the superstring a multi-loop regulator must be chosen to agree with the modular properties of the IIB superstring S-matrix.

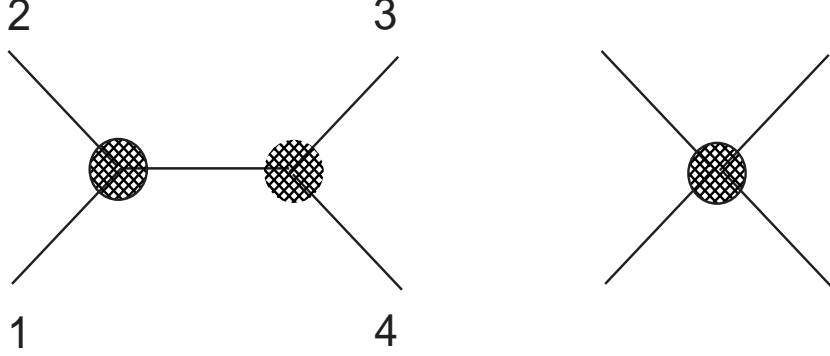


Figure 4: The circles denote quantum corrected effective vertices derived from an effective action. Modular invariance is regained after sewing to obtain trees.

4 Modular Forms: $SL(2, Z)$ and Eisenstein Functions

In this section we describe the generalized Eisenstein series on the fundamental domain of $U(1) \backslash SL(2, R) / SL(2, Z)$ and their relevant properties. These functions are non-holomorphic modular forms defined by

$$E_s^{(q, -q)}(\tau, \bar{\tau}) = \sum_{(m, n) \neq (0, 0)} \frac{\tau_2^s}{(m + n\tau)^{s+q} (m + n\bar{\tau})^{s-q}} , \quad (4.42)$$

where the sum over the integral pairs (m, n) does not include $m = n = 0$. For $q = 0$ we recover the non-holomorphic Eisenstein series. The series converges for $s > 1$ on the fundamental domain ($|\tau_1| \leq 1/2$ and $|\tau| \geq 1$) and transforms under the fractional linear transformation

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d} , \quad (4.43)$$

as

$$E_s^{(q, -q)}(\tau', \bar{\tau}') = \left[\frac{c\tau + d}{c\bar{\tau} + d} \right]^q E_s^{(q, -q)}(\tau, \bar{\tau}) . \quad (4.44)$$

Although the Eisenstein series are modular invariant, the generalized series for $q \neq 0$ transforms with a weight $(q, -q)$. The latter may be used to find additional modular invariant functions by pairing them together as in $E_s^{(q, -q)}(\tau, \bar{\tau}) E_s^{(-q, q)}(\tau, \bar{\tau})$ together with additional n -tuple products where the weights add up as $\sum_{i=1}^n q_i = 0$.

The generalized Eisenstein series $E_s^{(q, -q)}(\tau, \bar{\tau})$ are related differentially to $E_s(\tau, \bar{\tau})$ by

$$E_s^{(q+1, -q-1)}(\tau, \bar{\tau}) = \left(i\tau_2 \frac{\partial}{\partial \tau} + \frac{q}{2} \right) E_s^{(q, -q)} , \quad (4.45)$$

and

$$E_s^{(q-1, q+1)}(\tau, \bar{\tau}) = \left(-i\tau_2 \frac{\partial}{\partial \bar{\tau}} - i\frac{q}{2}\right) E_s^{(q, -q)} . \quad (4.46)$$

The asymptotic expansion for $q \neq 0$ may be obtained from $q = 0$ via the relation in (4.45).

Further, the functions satisfy the $SL(2, Z)$ covariantized differential relation,

$$4\left(\tau_2 \frac{\partial}{\partial \tau} + i\frac{1-q}{2}\right)\left(\tau_2 \frac{\partial}{\partial \bar{\tau}} - i\frac{q}{2}\right) E_s^{(q, -q)}(\tau, \bar{\tau}) = \lambda_s^{(q, -q)} E_s^{(q, -q)}(\tau, \bar{\tau}) , \quad (4.47)$$

with an iteratively constructed eigenvalue from (4.45).

The complete asymptotic form for large τ_2 is found via manipulating the series through a Poisson resummation. The general form is

$$E_s^{(0,0)}(\tau, \bar{\tau}) = a_s \tau_2^s + b_s \tau_2^{1-s} \quad (4.48)$$

$$+ \frac{2\sqrt{\tau_2}\pi^s}{\Gamma(s)} \sum_{(m,n) \neq (0,0)} \left|\frac{m}{n}\right|^{s-1/2} K_{s-\frac{1}{2}}(2\pi\tau_2|mn|) e^{2\pi i mn \tau_1} , \quad (4.49)$$

where $K_s(x)$ is the standard modified Bessel function with expansion for large x ,

$$K_r(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \left[\sum_{n=0}^{\infty} \frac{1}{(2x)^n} \frac{\Gamma(r+n+\frac{1}{2})}{\Gamma(r-n+\frac{1}{2})\Gamma(n+1)} \right] \quad (4.50)$$

and where the coefficients in (4.49) are,

$$a_s = 2\zeta(2s) \quad b_s = 2\sqrt{\pi}\zeta(2s-1) \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} . \quad (4.51)$$

The asymptotic form for all $s > 1$ contains two terms which are powers of τ_2 together with an infinite series of exponentially suppressed terms. The former will relate to the perturbative expansion of the superstring S-matrix and the latter to a series of conjectured D-instanton corrections in the uncompactified theory. The generalized Eisenstein functions also have the same structure; however, the coefficients are different. For example,

$$a_s^{(1,-1)} = 2s\zeta(2s) \quad b_s^{(1,-1)} = 2\sqrt{\pi}(1-s)\zeta(2s-1) \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} . \quad (4.52)$$

All of these modular forms for general q -values give rise to two power suppressed terms in the large τ_2 limit together with exponentially suppressed corrections.

Table 3: The U-duality structure in toroidally compactified IIB superstring theory. The Z_2 in $d = 9$ reflects the element that takes the IIB string into the IIA.

	Non-perturbative U-Duality Group
$d = 10$	$SL(2, Z)$
$d = 9$	$SL(2, Z)(\times Z_2)$
$d = 8$	$SL(2, Z) \times SL(3, Z) = E_{3(3)}(Z)$
$d = 7$	$SL(5, Z) = E_{4(4)}(Z)$
$d = 6$	$SO(5, 5, Z) = E_{5(5)}(Z)$
$d = 5$	$E_{6(6)}(Z)$
$d = 4$	$E_{7(7)}(Z)$

5 U-dualities and Constraints

In order to examine the same construction of the S-matrix in dimensions other than ten and compare with maximally extended supergravity theories, we shall compactify IIB superstring theory on $M_d \times T^{10-d}$. In this section, we examine the modular form construction and find the similar truncation property to $d = 10$ in perturbation theory.

The S-duality inherited in ten dimensions is modified to the discrete U-duality group acting on all of the moduli of the toroidally compactified theory [14], and the low-energy theory is described by the dimensional reduction of the $d = 10$ IIB supergravity. The non-perturbative vacuum is parameterized by the full complement of scalar fields in the lower dimensional maximally extended supergravity. In this section, we generalize the previous construction for the S-matrix of ten-dimensional string theory to the compactified cases with the use of additional automorphic functions.

The vacuum state of compactified supergravity is parameterized by the values of scalars living in the symmetric space $G(R)/H(R)$ where H is the maximal compact subgroup of G ; we shall take the group $G(R) = E_{p+1(p+1)}(R)$ which contains as a subgroup $SO(p, p, R)$ (with the latter having maximal compact subgroup $SO(p, R) \times SO(p, R)$). Duality transformations generate an equivalence class of theories identified by an action of the infinite discrete group $G(Z)$. In compactifications $M_d \times T^{10-d}$ there is a T-duality group $SO(d, d, Z)$ and a non-perturbative action of $SL(2, Z)$ (the coupling constant of the dilaton and axion scalar of the uncompactified IIB superstring) inherited from the uncompactified ten-dimensional IIB superstring. The full symmetry of the equations of motion and the quantum U-duality group is known to be larger and is listed in table 5 [32]. (We will not consider the $O(d, d, Z)$ enlargement of the T-duality group as the elements with negative determinant exchange the IIB string with the type IIA one.) The larger U-duality group $E_{11-d(11-d)}(Z)$ ($d \leq 9$) contains the subgroup $SO(10 - d, 10 - d, Z) \times SL(2, Z)$ for different d . It treats all of the scalars in the supergravity on the same footing, although the dilaton plays a special role in

the compactified string in measuring the loop expansion. The general form of the symmetry and the constraints on the functional form of automorphic terms in the low-energy expansion of the S-matrix appear to fix the functional form of scattering in M-theory, defined in this case by T-dualizing the result of graviton scattering in IIB on S_R and decompactifying the two-torus with complex structure τ .

The simplest theory from the above is uncompactified type IIB superstring theory which is parameterized by $U(1)\backslash SL(2, R)$. Duality transformations take the theories and maps them to equivalent theories under $SL(2, Z)$; the fundamental domain of $U(1)\backslash SL(2, R)/SL(2, Z)$ parameterizes inequivalent vacua of the ten-dimensional IIB superstring. In the compactified theories the inequivalent theories are described by vacuum expectation values of the set of moduli parameterizing the fundamental domain of $H(R)\backslash E_{11-d(11-d)}(R)/E_{11-d(11-d)}(Z)$. In analogy with the ten-dimensional case we consider as building blocks of the S-matrix the $E_{11-d(11-d)}(Z)$ invariant functions which satisfy the differential equations

$$\nabla_{H(R)\backslash E_{11-d(11-d)}(R)} f_s(\phi_j) = \lambda_{11-d,s} f_s(\phi_j) , \quad (5.53)$$

and their generalizations, in analogy with the Eisenstein series for the ten-dimensional uncompactified superstring theory. This form immediately decompactifies to higher dimensional U-duality invariant Eisenstein series differentially because the metric reduces on the U-duality subgroup. These functions relevant for the lowest derivative term, i.e. R^4 , have been discussed for $d \geq 7$ in [34] and for general integer dimensions in [35]. In the latter work the unified set of perturbative and non-perturbative contributions to the R^4 term was argued to be described by

$$A_d = \frac{1}{\kappa_d^2} \int d^d x \sqrt{g} f_d(\phi_j) R^4 , \quad (5.54)$$

where,

$$f_d(\phi_j) = E_{\text{string}; s=3/2}^{E_{11-d(11-d)}(Z)}(\phi_j) , \quad (5.55)$$

where κ_d^2 is the gravitational coupling constant in $10 - d$ dimensions and the Eisenstein function takes into account the summation over the “string multiplets”; the string multiplet is a unified representation of the full duality group $E_{11-d(11-d)}(Z)$ of the particle and membrane representations with given charges under the T-duality group and we refer the reader to [35] for the construction of the general invariants (5.55). This function gives rise to perturbative contributions at tree-level and one-loop only which is in accord with the explicit results of maximal supergravity in d dimensions in [15]. It also limits consistently to reproduce decompactified dimensions as well as giving agreement with the known tree- and one-loop level contributions. The perturbative truncation property of the function in (5.55) permits contributions at only tree and one-loop level and is consistent with the tensor properties of maximally extended supergravity in different integer dimensions [15].

The general constrained Eisenstein functions for a symmetric space $G(R)/H(R)$ are described in detail in [35], and are defined for a given representation \mathcal{R} of G by,

$$E_{R,s}^{G(Z)}(\phi_j) = \sum_{m \in \Lambda_R \neq 0} \frac{\delta(m \wedge m)}{[M^2(m)]^s} . \quad (5.56)$$

The form in (5.56) is analogous to constructing modular invariant functions on the fundamental domain of $SL(2, Z)$ in the manner of

$$g(\tau) = \sum_{\gamma \in G} f(\gamma \cdot \tau) , \quad (5.57)$$

but with additional scalars involved in the mass formula in the denominator of (5.56).

In (5.56), ϕ_j denotes elements in $G(R)/H(R)$ and m is a vector in the integer lattice Λ_R transforming in the representation R . The $m \wedge m = 0$ condition projects onto sets of integers m so that the symmetric tensor product $R \times_s R$ gives its highest irreducible multiplet which defines m to be the most symmetric piece of the direct product. Physically, the set of integers in the lattice Λ_R labels the set of BPS states in the representation R of the duality group (for example, one may consider the unified string representation of $E_{11-d(11-d)}(Z)$ or its further decompositions.) The condition $m \wedge m = 0$ is the half BPS condition, and the mass formula contributing to the sum in the Eisenstein series for sets of states contributing to (5.56) is

$$M_{\text{BPS}}^2(m) = m \cdot M \cdot m . \quad (5.58)$$

One could relax the condition $m \wedge m = 0$ to include quarter and eighth BPS states by considering a more general definition of the constrained Eisenstein function in (5.56). The functions $E_R^{G(Z)}(\phi_j)$ are by construction invariant under the duality group when $G(Z) = E_{11-d(11-d)}(Z)$ and take values in the fundamental domain $H(R) \backslash E_{11-d(11-d)}(R) / E_{11-d(11-d)}(Z)$. The various discrete U-duality groups are listed in table 4.

In analogy with the generalized Eisenstein functions on the $U(1) \backslash SL(2, R) / SL(2, Z)$ fundamental domain, a generalized series $E_{R,s}^{(q,-q), E_{11-d(11-d)}(Z)}(\tau, \bar{\tau})$ in the limit of zero moduli may be given. The asymptotic form in (5.70) may be given a weight by covariantizing as in (4.45). The coupling $\tau_2 V_{10-d}^{\frac{4}{10-d}}$ is inert under $SL(2, Z)$ transformations.

The simplest forms in the d -dimensional set are the Eisenstein functions relevant to uncompactified IIB superstring theory. The construction given in previous sections generalizes similarly, although the precise functional form is more complicated, to the toroidally compactified cases; however, the $E_{11-d(11-d)}(Z)$ Eisenstein functions relevant to lower target dimensions have different convergence properties but the same type of perturbative truncation to a finite order in the coupling constants as is found in later sections in the small volume and null moduli limit where only the action on the $SL(2, Z)$ transformations is taken on the string coupling. As a consistency in the decompactification limit where the radii are taken

Table 4: “String” multiplets of $E_{11-d(11-d)}$ relevant to compactified $M_d \times T^{10-d}$ IIB superstring theory and their decompositions into $SL(11-d, Z)$ and T-duality $SO(10-d, 10-d)$ groups.

d	U-Duality Group	string rep	$SL(11-d)$ rep	$SO(10-d, 10-d)$ rep
10	$SL(2, Z)$	1	1	1
9	$SL(2, Z)$	2	2	1 + 1
8	$SL(2, Z) \times SL(3, Z)$	(1,3)	3	1 + 2
7	$SL(5, Z)$	5	4+1	4 + 1
6	$SO(5, 5, Z)$	10	5+5	1 + 8_S + 1
5	$E_{6(6)}(Z)$	27	6 + 15 + $\bar{6}$	1 + 10 + 16
4	$E_{7(7)}(Z)$	133	7 + 28 + 35 + ...	1 + (1+ 66) + 32 + ...

to infinity, the successively higher-dimensional modular ansätze for the S-matrix should be retrieved.

In order to generalize the S-matrix form in dimensions $d \geq 7$ analogous to the one in ten dimensions, we need a systematic treatment of the Eisenstein series only for the $SL(p \leq 5)$ groups, while in lower dimensions the exceptional groups enter as special cases with representations that may be decomposed into the $SL(p)$ ones or the T-duality ones.

The symmetry of the supergravity field equations pertaining to the U-duality group enforce that the different states and solitonic configurations from the compactified string fall into representations of the duality group [14]. These representations correspond to half-BPS states and are used in the construction (5.56) to find invariant functions under $E_{11-d(11-d)}(Z)$ which live on the fundamental domain parameterizing the moduli space of the compactified IIB superstring theory (listed in Table 3). As discussed in [35], these representations are listed in tables 4-6, and there may be degeneracy amongst the different constrained Eisenstein series between the functions constructed from the different representations at lowest order in derivatives, R^4 . Consistency requires the higher-derivative terms in the low-energy limit to reduce under decompactification to the higher-dimensional forms (in integer dimensions).

In the remainder of the section we confirm the τ_2 dependence in Einstein frame of the perturbative series in different integral dimensions through comparison with the expansions of the appropriate modular forms at small coupling. The agreement of the expansions with the powers of τ_2 is a check on the modular properties of the scattering in different dimensions.

The transformation of string frame to Einstein frame is given by

$$g_{\mu\nu}^{(s)} = e^{\alpha\phi} g_{\mu\nu}^{(E)} , \quad (5.59)$$

together with

$$\sqrt{g_{(s)}} = e^{\alpha\frac{d}{2}\phi} \sqrt{g_{(E)}} \quad R_{(s)} = e^{-\alpha\phi} R_{(E)} . \quad (5.60)$$

Table 5: “Particle” multiplets of $E_{11-d(11-d)}$ relevant to compactified $M_d \times T^{10-d}$ IIB superstring theory and their decompositions into the fundamental SL and T-duality $SO(10-d, 10-d)$ groups.

d	U-Duality Group	particle rep	SL rep	SO rep
10	$SL(2, Z)$	1	1	1
9	$SL(2, Z)(\times Z_2)$	3	1 + 2	1 + 2
8	$SL(2, Z) \times SL(3, Z)$	(2, 3)	3 + $\bar{3}$	2 + 4
7	$SL(5, Z)$	10	4 + 6	4 + 6
6	$SO(5, 5, Z)$	16	5 + 10 + 1	$8_v + 8_c$
5	$E_{6(6)}(Z)$	27	6 + 15 + $\bar{6}$	1 + 10 + 16
4	$E_{7(7)}(Z)$	56	7 + 21 + $\bar{7}$ + 21	12 + 32 + 12

The Einstein frame of the Einstein-Hilbert action derived from the string frame is found from,

$$\alpha = \frac{4}{d-2} , \quad (5.61)$$

or in terms of the ten-dimensional string coupling $g_s^{(10)}$

$$g_{\mu\nu}^{(s)} = \tau_2^{-\frac{4}{d-2}} g_{\mu\nu}^{(E)} \quad s_{ij}^{(s)} = \tau_2^{\frac{4}{d-2}} s_{ij}^{(E)} , \quad (5.62)$$

where we have illustrated how invariants scale also in (5.62).

In (5.62) we see that dimensions $d < 6$ is special: The Einstein frame coupling dependence increases by τ_2^β with $\beta > 1$ for every pair of derivatives in the low-energy expansion and the transition dimension is $d = 6$. Six dimensions is also singled out due to the presence of the four-form self-dual moduli. At weak IIB coupling, $\tau_2 \rightarrow \infty$, the dimensions $d \leq 6$ have the feature that the tree-level terms are larger in string coupling at higher derivatives (in $d > 6$ the terms are of smaller value), although the appropriate expansion parameter in this comparison at low-energy is

$$\alpha = p_{(E)}^2 \tau_2^{\frac{4}{d-2}} , \quad (5.63)$$

which may give a decreasing effect if the momentum scale is such that $\alpha < 1$. (The volume modulus is given in (5.65)). The coupling constant dependence in the low-energy S-matrix expansion has the distinguishing feature in $d \leq 6$, the same dimensions in which supergravity has extra finiteness properties in the Regge limit and also to higher loop orders implied by the automorphic IIB graviton scattering.

Table 6: “Membrane” multiplets of $E_{11-d(11-d)}$ relevant to compactified $M_d \times T^{10-d}$ IIB superstring theory and their decompositions into the mapping class $SL(10-d)$ and T-duality $SO(10-d, 10-d)$ groups.

d	U-Duality Group	memb. rep	SL rep	SO rep
10	$SL(2, Z)$	1	1	1
9	$SL(2, Z)$	1	1	1
8	$SL(2, Z) \times SL(3, Z)$	(2,1)	1+1	2
7	$SL(5, Z)$	5	4+1	4 + 1
6	$SO(5, 5, Z)$	16	5+10+ 1	$8_v + 8_c$
5	$E_{6(6)}(Z)$	78	1 + 20 + 36 + ...	16 + 16 + (1+45)
4	$E_{7(7)}(Z)$	912	1 + 35+...	32 + ...

Alternatively, we may transform to Einstein frame in ten dimensions and then compactify on an T^{10-d} torus, taking into account the volume dependence of the torii. The frame dependence in (5.62) forces the perturbative series in Einstein frame of the compactified IIB superstring to have tree-level contributions to $\square^k R^4$ with the τ_2 dependence,

$$A_4^{\text{tree},k} = N_k \left[\tau_2 V_{10-d}^{\frac{4}{10-d}} \right]^{\left(\frac{3+k}{d-2}\right)\left(\frac{10-d}{2}\right)} \tau_2^{\frac{3}{2}+\frac{k}{2}} \int d^d x \sqrt{g_E} \square_{(E)}^k R_{(E)}^4 \quad (5.64)$$

$$= N_k V_{10-d}^{\frac{2(3+k)}{(d-2)}} \tau_2^{\frac{12+4k}{d-2}} \int d^d x \sqrt{g_E} \square_{(E)}^k R_{(E)}^4, \quad (5.65)$$

where the volume factor $\tau_2 V_{10-d}^{\frac{4}{10-d}}$ is inert under the subgroup $SL(2, Z)$ transformations of the U-duality group in Einstein frame. In $d = 10$ the perturbative series reproduces the series in (2.20) and sub-leading string genus corrections to (5.65) generate powers τ_2^{-2g} relative to the tree-level; the coefficient is modified in the toroidal compactifications due to the volume dependence and scaling in d -dimensional compactified Einstein frame. We collect some explicit powers of τ_2 dependence of the $\square^k R^4$ terms in the low-energy expansion of the IIB superstring S-matrix on $M_d \times T^{10-d}$ dimensions below:

The τ_2 dependence in (5.65) at tree-level in Einstein frame may be matched with the modular form expansions in $d \leq 10$ in accord with criteria (1) in section 2. The Eisenstein function in lower dimensions on the appropriate fundamental domain with value $s = 3/2$ and $s = 5/2$ gives rise to the dependence in Table 7.

5.1 Asymptotic Limits

We give the large coupling expansions in this subsection and first examine the simplest case, namely the $SL(n, Z)$ Eisenstein functions to the fundamental (and anti-fundamental)

Table 7: Example couplings in τ_2^β at genus zero in Einstein frame.

d	R^4	$\square R^4$	$\square^2 R^4$	\dots
10	$\frac{3}{2}$	2	$\frac{5}{2}$	
9	$\frac{12}{7}$	$\frac{16}{7}$	$\frac{20}{7}$	
8	2	$\frac{10}{3}$	$\frac{14}{3}$	
7	$\frac{12}{5}$	$\frac{16}{5}$	4	
6	3	4	5	
5	4	$\frac{16}{3}$	$\frac{20}{3}$	
4	6	8	10	

representations. The $SL(2, Z)$ Eisenstein function to the fundamental representation is given by

$$E_s(\tau, \bar{\tau}) = \sum_{(p,q) \neq (0,0)} \frac{\tau_2^s}{|p + q\tau|^s}, \quad (5.66)$$

and for $SL(n, Z)$ in the fundamental representation we have the explicit analogous function,

$$E_{R=d,s}^{\text{SL}(n,Z)}(\phi_j) = \sum_{(m^1, \dots, m^n) \neq 0} \frac{1}{[m^i g_{ij}(\phi_k) m^j]^s}. \quad (5.67)$$

The anti-fundamental representation is given by (5.67) but with indices lowered/raised. The series in (5.67) is absolutely convergent only for $s > n/2$.

Upon taking a d -dimensional torus into a direct product of one circle of radius R times a T^{d-1} generically non-orthogonal torus together with large R , the limit of (5.67) is

$$E_{R=d,s}^{\text{SL}(d,Z)}(\phi_j) = E_{R=d-1,s}^{\text{SL}(d-1,Z)}(\tilde{\phi}_j) + \frac{2\pi^s \Gamma(s - \frac{d-1}{2}) \zeta(2s - d + 1)}{\pi^{s - \frac{d-1}{2}} \Gamma(s) R^{2s-d+1} V_{d-1}} \quad (5.68)$$

$$+ \frac{2\pi^s}{\Gamma(s) R^{2s-d-1}} \sum_{m^a, n^b} \left| \frac{n^a g_{ab} n^b}{m^2} \right| K_{s - \frac{d-1}{2}} \left(2\pi |m| R \sqrt{n^a g_{ab} n^b} \right). \quad (5.69)$$

The asymptotic limit in (5.69) of $E_{R=d,s}^{\text{SL}(d,Z)}(\phi_j)$ gives rise to the Eisenstein series of $SL(d-1)$ to the fundamental representation together with an infinite number of terms depending on the large radius (one power-behaved and the rest exponentially suppressed) which vanish as $R \rightarrow \infty$. Although only the fundamental representation is analyzed in (5.69) a similar structure is expected for other representations of the duality group $U(Z)$; the zero moduli

approximation also generates this limit. This structure is consistent with the expected form of the modular properties of the S-matrix in the decompactification limit and the exponential terms in the expansion of the forms represent the various wrapping of string solitonic states in the compact directions of the compactified torii dimensions. In subsequent work we shall analyze the d -dimensional form in view of the finite radii dependence of the toroidal compactifications.

The weak coupling expansion is defined by taking the dilaton IIB scalar to large vacuum expectation value (small string g_s coupling constant). Turning off all of the vacuum values of the scalar fields except for τ which is acted on by the S-duality $SL(2, Z)$ subgroup of $U(Z)$, these (regulated) functions reduce to the Eisenstein series for $E_s^{SL(2, Z)}(\tau, \bar{\tau})$ together with a perturbative truncation property described in previous sections that gives rise to the same power counting in the low-energy limit of the superstring scattering elements.

We can consider the special point of the moduli of the toroidally compactified theory when all moduli except for the ten-dimensional dilaton have vanishing values. In this case, the large τ_2 behavior of the Eisenstein series in (5.53) may be constructed by solving the differential equation for the $SL(p, Z)$ representation decompositions into the $SL(2, Z)$ subgroup of the various string multiplet contributions. The functional form of the small moduli contribution is invariant under $SL(2, Z)$ transformations and is,

$$E_s^{E_{11-d(11-d)}}(\tau, \bar{\tau}) = \left[\tau_2 V_{10-d}^{\frac{4}{10-d}} \right]^{(\frac{3+k}{d-2})(\frac{10-d}{2})} \sum_{(p,q) \neq (0,0)} \frac{\tau_2^s}{|p + q\tau|^{2s}} + \dots, \quad (5.70)$$

for $k = 2s - 3$, which truncates in every dimension, as it does in $d = 10$ (for the subtracted or convergent series E_s). The form in (5.70) is due to the $SL(2, Z)$ subgroup of the larger U-duality group. Constructions using the generalized Eisenstein series follows similarly in the toroidal compactifications, with the difference involving the volume factor of the $10 - d$ dimensional tori. The unitarity construction described in previous sections guarantees that the imaginary parts will also be invariant under the U-duality group, after construction of the supersymmetric extension of the $\square^k R^4$ terms and integrating to find the imaginary parts.

5.2 D=8 Example

In the remainder of this section we give an example for the polynomial terms in the low-energy expansion of the S-matrix in the compactified theories analogous to the one presented for the uncompactified IIB superstring but invariant under the larger U-duality group. We give the example of the $D = 8$ theory which has a U-duality group in the toroidally compactified theory of $SL(2, Z) \times SL(3, Z)$ and a set of scalars parameterizing the fundamental domain of $SO(2, R) \times SO(3, R) \setminus SL(2, R) \times SL(3, R) / SL(2, Z) \times SL(3, Z)$. An explicit parameterization of the moduli space consists of the complex structure of the two-torus with metric (and

volume $V = V_2$),

$$g_{ab} = \frac{V}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix} \quad (5.71)$$

where U spans the region $U(1) \backslash SL(2, R)$ and with the enhanced moduli of $SL(3, R)/SO(3)$. The scalar $\tau_2 V$ is invariant under the $SL(2, Z)$ sub-group of $SL(3, Z)$ acting on the IIB scalar τ . The moduli of $SL(3, R)/SO(3)$ consist of the dilatonic and axionic couplings τ together with those of the reduction of the anti-symmetric complex tensors $B_{NS,R} = \epsilon_{ab} B_{NS,R}^{ab}$ grouped as $B = B_R + \tau B_{NS}$ into the symmetric coset matrix form with the matrix satisfying the determinant $\det M = 1$ condition,

$$M = (\tau_2 V)^{-1/3} \begin{pmatrix} \frac{1}{\tau_2} & \frac{\tau_1}{\tau_2} & \frac{\mathcal{R}(B)}{\tau_2} \\ \frac{\tau_1}{\tau_2} & \frac{|\tau|^2}{\tau_2} & \frac{\mathcal{R}(\bar{\tau}B)}{\tau_2} \\ \frac{\mathcal{R}(B)}{\tau_2} & \frac{\mathcal{R}(\bar{\tau}B)}{\tau_2} & \tau_2 V^2 + \frac{|B|^2}{\tau_2} \end{pmatrix}. \quad (5.72)$$

The matrix M in (5.72) together with U parameterizes the scalar manifold $U(1) \backslash SL(2, R) \times SL(3, Z)/SO(3, R)$ of the classical IIB superstring theory compactified on a two-torus with metric in (5.71) and the complex structure parameterized by U . The action of the scalars and Einstein-Hilbert low-energy theory is

$$S = \frac{1}{\kappa_8^2} \int d^8 x \sqrt{g} \left[R - \frac{\partial U \partial \bar{U}}{U_2^2} + \frac{1}{4} \text{Tr}(\partial M \partial M^{-1}) \right], \quad (5.73)$$

and we are primarily interested in this work in the string corrections to gravitational action. The Eisenstein series relevant to the eight-derivative term has been elucidated for the R^4 term in [34, 35].

The R^4 term in Einstein frame arising in the low-energy expansion is described in (5.55). In accord with the functional form in (2.3) and (2.13) we take the twelve derivative term to be

$$S_{d=8}^{12} = \int d^8 x \sqrt{g} E_{\text{string}; s=5/2}^{SL(2,Z) \times SL(3,Z)}(\phi_j) \square^2 R^4, \quad (5.74)$$

which generalizes to arbitrary dimensions to

$$S_d^{12} = \int d^d x \sqrt{g} E_{\text{string}; s=5/2}^{E_{11-d(11-d)}(Z)}(\phi_j) \square^2 R^4. \quad (5.75)$$

The string multiplet in $d = 8$ decomposes under $SL(2, R) \times SL(3, Z)$ into the representation $\mathbf{1} \times \mathbf{3}$, and accordingly, the automorphic contribution in (5.74) breaks into,

$$S_{d=8}^{12} = \int d^8 x \sqrt{g} E_{\mathbf{3}; 5/2}^{SL(3,Z)}(\phi_j^{(2)}) \square^2 R^4, \quad (5.76)$$

or alternatively,

$$S_{d=8}^{12} = \int d^8x \sqrt{g} \left[E_{\mathbf{1}; \tilde{s}_1}^{SO(2,2,Z)}(\phi_j^{(1)}) + E_{\mathbf{2}; \tilde{s}_2}^{SO(2,2,Z)}(\phi_j^{(2)}) \right] \square^2 R^4 . \quad (5.77)$$

The modular weights $s_1 = 3/2$ and $s_2 = 5/2$ are determined by decomposing the form of the modular summand into the lower representations. The complex structure of the torus U parameterizes the $SL(2, R)/U(1)$ portion of the T-duality group ($SO(2, 2, R) = SL(2, R) \times SL(2, R)$) and in the decompactification limit.

The $SL(3, Z)$ Eisenstein functions entering into (5.76) are explicitly in the fundamental representation constructed from the matrix in (5.72),

$$\begin{aligned} E_{\mathbf{3}; \mathbf{s}}^{SL(3,Z)}(\phi_j^{(2)}) &= \sum_{a,b=1,2,3} \sum_{n^a, n^b} \frac{1}{[n^a M_{ab} n^b]^s} \\ &= \sum_{(n^1, n^2, n^3) \neq (0,0,0)} \tau_2^s \frac{(\tau_2 V)^{s/3}}{[|n^1 + n^2 \tau + n^3 B|^2 + (n^3 \tau_2 V)^2]^s} , \end{aligned} \quad (5.78)$$

together with the (subtracted) form of the order one Eisenstein function of the complex structure U ,

$$\hat{E}_{\mathbf{1}; \mathbf{1}}^{SL(2,Z)}(\phi_j^{(1)}) = -\pi \ln \left(U |\eta(U)|^4 \right) , \quad (5.79)$$

that we must also add in. The series in (5.78) is explicitly invariant under the $SL(3, Z)$ transformations and to leading order in τ_2 has the expected power dependence to agree with the Einstein frame string coupling at tree-level (as it must if the compactified string theory possesses a U-duality structure). At small volume and at zero B moduli we have,

$$\begin{aligned} E_{\mathbf{3}; \frac{5}{2}}^{SL(3,Z)}(\phi_j^{(2)}) &= \tau_2^{\frac{10}{3}} V^{\frac{5}{3}} \sum_{(m,n) \neq (0,0)} \frac{1}{|m + n\tau|^5} \\ &= \tau_2^{\frac{5}{6}} V^{\frac{5}{3}} \left(\tau_2^{\frac{5}{2}} \zeta(5) + a \tau_2^{-\frac{3}{2}} + \mathcal{O}(e^{2\pi i \tau}) \right) . \end{aligned} \quad (5.80)$$

The perturbative series in (5.80) only receives contributions at genus zero and two in this limit.

In the decompactification limit $R_1, R_2 \rightarrow \infty$, the functional form in (5.74) must reconcile with the $d = 9$ and $d = 10$ results for the twelve-derivative term and be invariant under the U-duality group $SL(2, Z)$. Using (5.69), we see that the only term surviving the large R limit is the $SL(3, Z)$ Eisenstein function in (5.76), which correctly limits to an order $s = 5/2$ $SL(2, Z)$ Eisenstein function. It would be interesting to check the finite radius dependence further.

6 Perturbative truncations in integral dimensions

In this section we examine the implications of the perturbative truncation, that of a maximum genus for individual terms in the derivative expansion, with maximally extended supergravity theories. The modular structure of the graviton scattering amplitude indicates the perturbative truncation property in different dimensions and implies a similar behavior in supergravity. This property is present already at order $\square^2 R^4$ [5, 16]. Two options to explain the perturbative truncation implied by the modular structure are either an infinite number of independent cancellations at each genus between the massive and massless modes are nullifying the contributions to $\square^k R^4$ for $k \geq 2$, or that the massive modes give rise to order α' higher corrections compared with the supergravity modes in the low-energy limit (as at $g = 0$ and $g = 1$) and that IIB supergravity possesses this truncation.

In order to compare the string amplitudes with the supergravity ones: the massive modes need to be distinguished from the massless ones in the $\alpha' \rightarrow 0$ limit at $g \geq 3$, the ghost dependence in the string amplitudes needs to be disentangled in the field theory limit, and regulator dependence in the supergravity theory implied by the integration region coming from the moduli space (at one-loop in (2.12)) has to be examined with regards to possible cancellations due to the region of integration. Ghost dependence in the genus two four-graviton scattering amplitude, for example, naively indicates a contribution to the R^4 term in the derivative expansion before moduli integration that is in disagreement with explicit two-loop supergravity graviton scattering [33]; consistency with S-duality and perturbative supergravity requires these dependencies to integrate to zero.

In the IIB supergravity limit of the superstring scattering, primitive divergences in $d = 10$ of the four-point amplitude are of the form:

$$A_4^{L=1,m=0} \sim \Lambda^2 R^4 + \square R^4 + \dots \quad (6.81)$$

$$A_4^{L=1,m \neq 0} \sim \square R^4 + \square^2 R^4 + \dots \quad (6.82)$$

The α' indicates an overall factor of \square or s_{ij} in comparison to the massless modes after they are normalized correctly at the given order.

At two-loops in field theory, an explicit twelve derivatives may be extracted from the loop integration and the amplitude has the generic tensor structure,

$$A_4^{L=2,m=0} \sim \square^2 (\Lambda^6 + \Lambda^4 \square + \Lambda^2 \square^2 + \square^3) R^4 + \dots \quad (6.83)$$

Explicit string theory calculations at genus two may be computed in the $\alpha' \rightarrow 0$ limit. If the massive modes of the string contribute at an order α' higher in the low-energy limit (as at order $g = 0$ and $g = 1$), then an additional pair of derivatives must also be extracted,

$$A_4^{L=2,m \neq 0} \sim \square^3 R^4 + \dots \quad (6.84)$$

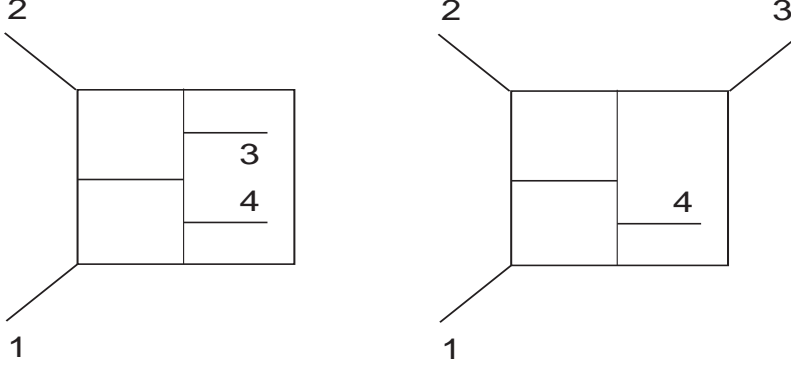


Figure 5: Example non-planar Feynman diagrams in supergravity contributing to the four-point function computed in [15] illustrated without the additional loop tensor within the integrand.

At tree and one-loop level the massive modes explicitly contribute an order higher in α' , and it is reasonable to suspect that such a property persists at higher order. However, an explicit integration of the genus two four-point function in the low-energy limit is required; picture dependence in this order complicates the analysis although unitarity might be faulted if this were not the case.

At three-loops the conjectured form [15] of the four-point supergravity amplitude in ten dimensions leads to

$$A_4^{L=3,m=0} \sim \square^2 \left(\Lambda^{14} + \Lambda^{12} \square + \dots \right) R^4 \quad (6.85)$$

and if the massive modes contribute to a higher-order in α' then again their contribution must be of an order higher in derivatives by dimensional analysis,

$$A_4^{L=3,m \neq 0} \sim \square^3 R^4 + \dots \quad (6.86)$$

The contribution to $\square^2 R^4$ in the low-energy limit of the IIB superstring amplitude gets contributions at most from genus two from the modular form $E_{5/2}(\tau, \bar{\tau})$ Eisenstein function; this requires the leading divergence in (6.85) to have a zero coefficient.

The agreement of the low-energy field theory modelling of the superstring requires an infinite number of cancellations or nullifications at higher genus for every \square^k term in the low-energy expansion. The genus zero and two contributions to the $\square^2 R^4$ term in the ten-dimensional S-matrix, for example, indicates that separately every contribution (primitive ones above for example) proportional to \square^2 at $g > 2$ has to be zero (every perturbative term at different genus differs by powers of τ_2^2). The same structure occurs because of the truncation $g_{\max}^k = \frac{1}{2}(k+2)$ and $g_{\min}^k = \frac{1}{2}(k+1)$, for even and odd k , for the higher derivatives.

The perturbative counting arising from the Eisenstein series, i.e. $g_{\min} = 0$ to g_{\max}^k , is most naturally explained in the field theory by an explicit extraction of an additional four derivatives at every order (as found for example at loop order one and two). In the Regge limit, where only the ladder diagrams contribute in the construction of [15], this property is seen to infinite loop order for the massless modes, and pure $N = 8$ supergravity at vanishing moduli is finite in this kinematical regime to all orders in four through six dimensions.

However, at three-loops it is not clear if an additional set of momenta may be extracted from the complete set of integral functions contributing to the amplitude due to possible non-planar contributions possessing only three- and higher particle cuts; the complete form of the non-planar contributions to the four-point supergravity amplitude are not explicitly known to this order and the tensor property of three and higher loops within the amplitude for the supergravity modes is an open question. Sample non-planar contributions are illustrated in Figure 5, where the internal powers of l^4 are not inserted [15].

The tensor property, i.e. an extraction of internal loop momenta associated with the gravitational couplings in the pattern $4(1 + L)$, implies that $N = 8$ supergravity with all moduli tuned to zero except for the dilaton coupling is finite in four, five and six dimensions.

The fact that the modular ansatz indicates such a cancellation in the different integer dimensions is suggestive of the tensor property, perhaps in a string-inspired regulator at higher loop order that preserves the S-duality structure in the supergravity quantum field theory. Such cancellations after integration could be associated with boundaries of the integration region and divergence contributions vanishing on them in different dimensions (for example, $\sum_{(p,q) \neq (0,0)} 1/|p + q\tau|^4$ on the fundamental domain at genus one). A complete three-loop supergravity divergence calculation is necessary to find out if there is a cancellation in the field theory (before or after integration), or an explicit two-loop string theory calculation to disentangle picture dependence in the field theory limit and verify the q expansion of the massive modes.

7 Conclusions

We have explored a manifestly S-duality (and U-duality) invariant perturbative series in the derivative expansion of four-graviton scattering in IIB superstring theory compatible with the perturbative structure. Generalizations to higher-point gravitational amplitudes are straightforward. We have written the derivative expansion in terms of combinations of generalized Eisenstein series with coefficients not generally fixed by tree-level IIB graviton scattering. In ten dimensions this form predicts the absence of singularities in the moduli space of vacua which is consistent with taking into account all non-perturbative corrections.

The duality structure imposes strong constraints on the perturbative series from the perturbative superstring amplitude calculations. Imposing the duality structure is an additional postulate on the S-matrix akin to unitarity in the approach of analytic S-matrix theory, and allows the exciting possibility of determining the functional form of graviton scattering in

IIB superstring theory. The form of the scattering amplitude is also useful for computing finite $\lambda = g^2 N$ effects of $N = 4$ super Yang-Mills correlation functions through the AdS/CFT correspondence [38, 39, 40].

Unitarity of the massless modes is $SL(2, Z)$ invariant by construction in this formulation. The thresholds associated with the massive modes of the string must be found by summing the derivative expansion. Furthermore, these thresholds at genus one and higher in IIB string perturbation theory are additional constraints not yet imposed on the full functional form of the derivative expansion in this work, and could be used to fix the coefficients of the linear combinations of the Eisenstein series. Factorization conditions and consistency of the D-instanton corrections with the coupling constant dependence of the instantons through a holographic AdS relation to $N = 4$ super Yang-Mills also provide additional constraints.

This approach, via T-duality from IIB theory on a circle, provides an avenue for pinning down in the derivative expansion the covariantized form of graviton scattering in the eleven-dimensional corner of M-theory. Furthermore, the same approach may be used to find the constraints of S-duality on the low-energy scattering obtained purely in $N = 4$ super Yang-Mills theory, the planar form of which in the unbroken theory has been conjectured in perturbation theory in terms of integral functions for the four-gluon scattering process [37].

Additional analysis through instanton calculus or constraints such as those listed above are necessary to determine uniqueness or possible contributions of cusp forms, for example, at higher derivatives which generate non-perturbative corrections. Further checks of S-duality at the amplitude level involve low-energy expansions of $g \geq 2$ genus four-graviton contributions.

The modular property in this work predicts a perturbative truncation, that of a maximum genus contribution for a given derivative order, when expanded in the string coupling constant beyond the known supergravity structure at two-loops. Although the direct field theory limit of the superstring is difficult to obtain explicitly for genus greater than two, the most straightforward interpretation given the α' expansion in the field theory limit is that the perturbative series of the massless modes also truncates, perhaps in one regulator that preserves the remnant of the S- and U-duality in the supergravity approximation or field equations. This feature in maximally extended supergravity theory indicates a much higher degree of finiteness in four through six dimensions than expected.

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References

- [1] M. B. Green and M. Gutperle, Nucl. Phys. **B498**, 195 (1997) [hep-th/9701093].
- [2] M. B. Green and P. Vanhove, Phys. Lett. **B408**, 122 (1997) [hep-th/9704145].
- [3] J. G. Russo and A. A. Tseytlin, Nucl. Phys. **B508**, 245 (1997) [hep-th/9707134].
- [4] M. B. Green, M. Gutperle and P. Vanhove, Phys. Lett. **B409**, 177 (1997) [hep-th/9706175].
- [5] M. B. Green, H. Kwon and P. Vanhove, hep-th/9910055.
- [6] N. Berkovits and C. Vafa, Nucl. Phys. **B533**, 181 (1998) [hep-th/9803145].
- [7] P. K. Townsend, Phys. Lett. **B409**, 131 (1997) [hep-th/9705160].
- [8] M. Cederwall and P. K. Townsend, JHEP **9709**, 003 (1997) [hep-th/9709002].
- [9] G. Chalmers and K. Schalm, JHEP **9910**, 016 (1999) [hep-th/9909087].
- [10] B. de Wit and D. Z. Freedman, Nucl. Phys. **B130**, 105 (1977).
- [11] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. **B76**, 409 (1978).
- [12] E. Cremmer and B. Julia, Phys. Lett. **B80**, 48 (1978).
- [13] B. de Wit and H. Nicolai, Nucl. Phys. **B208**, 323 (1982).
- [14] C. M. Hull and P. K. Townsend, Nucl. Phys. **B438**, 109 (1995) [hep-th/9410167].
- [15] Z. Bern, L. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, Nucl. Phys. **B530**, 401 (1998) [hep-th/9802162].
- [16] M. B. Green and P. Vanhove, hep-th/9910056.
- [17] J. G. Russo, Phys. Lett. **B417**, 253 (1998) [hep-th/9707241].
- [18] J. G. Russo, Nucl. Phys. **B535**, 116 (1998) [hep-th/9802090].
- [19] M.B. Green and G. Moore, unpublished.
- [20] M. B. Green, J. H. Schwarz and L. Brink, Nucl. Phys. **B198**, 474 (1982).
- [21] L. Bel, Acad. Sci. Paris, Comptes Rend. **247**:1094 (1958), **248**:1297 (1959); I. Robinson, unpublished.
- [22] E. D'Hoker and D. H. Phong, Nucl. Phys. **B440**, 24 (1995) [hep-th/9410152].
- [23] G. Chalmers, Nucl. Phys. **B524**, 295 (1998) [hep-th/9712129].

- [24] E. D'Hoker and D. H. Phong, Rev. Mod. Phys. **60**, 917 (1988).
- [25] M. B. Green and S. Sethi, Phys. Rev. **D59**, 046006 (1999) [hep-th/9808061].
- [26] O. Lechtenfeld and A. Parkes, Nucl. Phys. **B332**, 39 (1990).
- [27] J. A. Harvey and G. Moore, Phys. Rev. **D57**, 2323 (1998) [hep-th/9610237].
- [28] D. J. Gross and E. Witten, Nucl. Phys. **B277**, 1 (1986).
- [29] N. Sakai and Y. Tanii, Nucl. Phys. **B287**, 457 (1987).
- [30] A. Terras, Harmonic analysis on symmetric spaces and applications, I and II, Springer-Verlag, 1985.
- [31] P. S. Howe and P. C. West, Nucl. Phys. **B238**, 181 (1984).
- [32] E. Cremmer, LPTENS 81/18 *Lectures given at ICTP Spring School Supergravity, Trieste, Italy, Apr 22 - May 6, 1981.*
- [33] R. Iengo and C. Zhu, JHEP **9906**, 011 (1999) [hep-th/9905050].
- [34] E. Kiritsis and B. Pioline, Nucl. Phys. **B508**, 509 (1997) [hep-th/9707018].
- [35] N. A. Obers and B. Pioline, hep-th/9903113.
- [36] E. Kiritsis, N. A. Obers and B. Pioline, hep-th/0001083.
- [37] Z. Bern, J. S. Rozowsky and B. Yan, Phys. Lett. **B401**, 273 (1997) [hep-ph/9702424].
- [38] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [hep-th/9711200].
- [39] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. **B428**, 105 (1998) [hep-th/9802109].
- [40] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].